Assignment 2

(28 February, 2026)

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Course: Algebraic Topology II (KSM4E02) Instructor: Aritra Bhowmick

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Q1. (*Internal Hom*) Given two chain complexes $C_\bullet, D_\bullet$, their *internal hom* is defined as

$$[C, D]_n = \prod_{k \in \mathbb{Z}} \hom(X_k, Y_{k+n}).$$

Define $d_n : [C, D]_n \to [C, D]_{n-1}$ via the formula

$$d_n(h_k) = \left(\partial^D_{k+n} \circ h_k - (-1)^n h_{k-1} \circ \partial^C_k\right)$$

Verify the following.

a. $[C, D]_\bullet = ([C, D]_n, d_n)$ is a chain complex.

b. The n-cycles of $[C, D]_\bullet$ are precisely the degree-n chain maps $C_\bullet \to D_\bullet$.

c. The 0^{th} homology of $[C, D]_\bullet$ consists of the chain homotopy classes of chain maps $C_\bullet \to D_\bullet$.

Note: Why *internal* hom?! Consider the category Ab. The $\hom$-set $\hom_{\mathrm{Ab}}(X, Y)$ itself is naturally an Abelian group, and thus it is already *internal* to the category. Similarly, in Top, we can use the compact-open topology to make $\hom_{\mathrm{Top}}(X, Y)$ into a topological space. For Ch, we could treat $\hom_{\mathrm{Ch}}(C_\bullet, D_\bullet)$ as a chain complex concentrated at degree 0. The internal hom functor $[C_\bullet, \_]$ is right adjoint to the tensor functor $\_ \otimes C_\bullet$; this is essentially the categorical definition of internal hom. We shall later see the definition of the tensor product of chain complexes. $3 + 3 + 4 = 10$

Q2. (*Good Pair*) A pair (X, A) is called a *good pair* if $A \subset X$ is closed, and there exists an open neighborhood $A \subset U \subset X$ such that U deformation retract onto A. Given a good pair (X, A), consider the diagram

$$\begin{array}{ccccc}
H_n(X, A) & \longrightarrow & H_n(X, U) & \longleftarrow & H_n(X \setminus A, U \setminus A) \\
\downarrow q_\star & & \downarrow q_\star & & \downarrow q_\star \\
H_n(X/A, A/A) & \longrightarrow & H_n(X/A, U/A) & \longleftarrow & H_n(X/A \setminus A/A, U/A \setminus A/A)
\end{array}$$

where q is the quotient map that identifies A to a point. Verify that the diagram is commutative. Show that $q_\star : H_n(X, A) \to H_n(X/A, A/A)$ is an isomorphism.

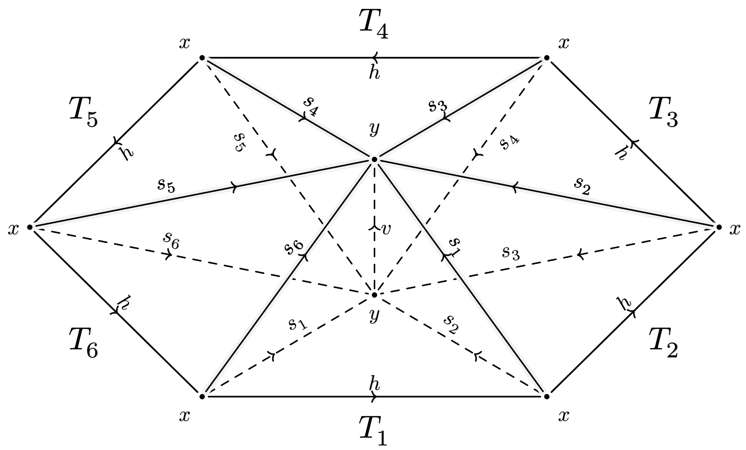
Hint : The green arrows are isomorphism by excision. For the blue arrow in the top row, note that $H_n(U, A) = 0$, and then use the long exact sequence for the triple (X, U, A); similar argument works for the bottom row as well. The right-most purple vertical arrow is induced by a homeomorphism $3 + 7 = 10$

Q3. (*Homology Long Exact Sequence of Cofibration*) Let $f : A \to X$ be a cofibration with the mapping cone C_f. Show that there exists a long exact sequence (in any homology theory)

$$\cdots \to \tilde{H}_{n+1}(C_f) \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(C_f) \to H_{n-1}(A) \to \cdots$$

Hint : Replace $f : A \to X$ by the mapping cylinder construction 10

Q4. (*Lens Space*) Construct a three-dimensional CW complex as follows. Take n-many 3-simplices $T_1, ..., T_n$. First, join them cyclically in the following way:



Next, identify the bottom face of T_i with the top face of T_{i+1}, where the indices are take $\bmod n$ (so that the bottom face of T_n is identified with the top face of T_1). Denote this space by L_n. Show that the singular homology groups are

$$H_i(L_n) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/n\mathbb{Z}, & i = 1 \\ 0, & i = 2 \\ \mathbb{Z}, & i = 3 \\ 0, & i \geq 4. \end{cases}$$

. This space is an example of a *lens space*. 10